WORKING PAPER
ALFRED P. SLOAN SCHOOL OF MANAGEMENT

ON INTERTEMPORAL PREFERENCES WITH A CONTINUOUS TIME DIMENSION II: THE CASE OF UNCERTAINTY

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Revised WP# 2105-89 July 1990

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March 1989
Revised July 1990

Abstract

We propose a family of topologies on the space of consumption patterns in continuous time under uncertainty. Preferences continuous in any of the proposed topologies treat consumptions at nearby dates as almost perfect substitutes except possibly at information surprises. The topological duals of the family of proposed topologies essentially contain processes that are the sums of processes of absolutely continuous paths and martingales. Thus if equilibrium prices for consumption come from the duals, consumptions at nearby dates have almost equal prices except possibly at information surprises. In particular, if the information structure is generated by a Brownian motion, the duals are composed of Itô processes. We investigate some implications of our topologies on standard models of choice in continuous time as well as on recent models of non time-separable representations of preferences. We also discuss the properties of prices of long-lived assets in economies populated with agents whose preferences are continuous in our topologies when there are no arbitrage opportunities.

*We are grateful to David Kreps and Robert Merton for helpful conversations, and to Phil Dybvig, a referee, whose comments improved the exposition. We would also like to thank John Heaton whose questions made us think more deeply about the relation between prices and the way information is revealed. Huang acknowledges research support under the Batterymarch Fellowship Program.

1 Introduction and Summary

Three factors determine the pattern of current prices for a consumption good to be delivered at different dates in the future. These are: the variations in supply as determined by endowment and technology, the evolution of information about future events, and preferences for consumption since consuming a commodity at one time is different from consuming the same commodity at another time. We focus in this paper on intertemporal preferences with two objectives in mind. First, we formalize the notion that a good consumed at one time is "close" to the same good consumed at a nearby time, except possibly when there is a significant change of information. Put differently, consumptions at nearby times are almost perfect substitutes except possibly at information surprises. Second, we show that, with preferences which exhibit this notion, current prices for consumption at two adjacent dates are very close except possibly at information surprises. This is a natural result since similar goods should have very close prices.

The intuition we formalize is certainly not new. We all experience effects of very recent consumption on current appetite. In addition, we also share the intuition that in an efficient market asset prices should not suddenly jump without some sudden change of expectations about future events, or, in other words, without a surprise. To explain this intuition, one could argue as follows. If one knows for sure that the price of a certain asset in a few moments will be much higher than it is now then one would be willing to delay his consumption for these few moments and use the proceeds to buy such an asset. Alternatively, one could borrow from someone else who is willing to delay his consumption, pay him a rate of interest and use the funds to realize the gains in the price of the asset. This behavior would force the price now to increase and become very close to the price of the asset a few moments from now.

Willingness to delay consumption hinges on the perception that consumptions at nearby dates are rather close. If preferences do not conform to this notion in the sense that no one is willing to delay consumption for a short while for a small fee, then a situation of a big jump in prices over short periods in the absence of any new information might prevail. In fact, this is a prediction of "standard" models in which preferences are represented by time-additive utility functions and close prices for consumptions at nearby dates are obtained mainly by exogenously specifying endowment processes that have continuous paths; see Duffie and Zame (1989) and Huang (1987).

This paper is the second part of a series of two papers. The first part, Huang and Kreps (1987), which we henceforth abbreviate as H&K, addresses the following questions: How might

one represent a consumption pattern in continuous time under certainty and what are the appropriate topologies on the space of consumption patterns that capture the idea that consumptions at nearby dates are almost perfect substitutes? Moreover, what form would the equilibrium prices take when individuals in the economy have preferences continuous in the appropriate topologies?

"Standard" answers exist for the questions raised by H&K: A consumption pattern on a time interval, say $[0,1]$, is represented by a real-valued function $c : [0,1] \to \Re_+$, where $c(t)$ is the consumption "rate" at time t. Two consumption patterns are close if they are close as functions in an L^p norm topology for some $1 \leq p < \infty$. An agent's preferences are represented by $\int_0^1 u(c(t),t)dt$, a time-additive functional of consumption rates, where, using the terminology of Arrow and Kurz (1970), $u(c,t)$ is a "felicity" function for consumption at time t. The equilibrium prices for consumption come from L^q with $\frac{1}{p}+\frac{1}{q}=1$, the topological dual of L^p; that is, the price at time zero of a consumption pattern c can be represented as $\int_0^1 c(t)y(t)dt$, where y is from L^q and $y(t)$ is the time zero price of one unit of consumption per unit time at time t.

H&K argue that these standard answers are unsatisfactory for the following reasons. First, in modeling consumption over time, one should allow consumption in gulps as well as at rates – observed consumption behavior such as having meals is in gulps. Second and more important, the L^p topology on the space of consumption rates is so strong along the time dimension that consumptions at nearby dates are perfect nonsubstitutes! As a consequence of the strong topology, the space of equilibrium prices is too rich. It includes, for example, discontinuous functions of time and continuous functions of unbounded variation signifying that equilibrium prices for consumptions at nearby dates either are very different or, even though are a continuous function of time, fluctuate in a nowhere differentiable manner.[1]

H&K represent a consumption pattern over time under certainty by an increasing[2] and positive function on $[0,1]$. Let x be such a function. Then $x(t)$ denotes the cumulative consumption from time zero to time t. If x is an absolutely continuous function of time, its derivatives exist almost everywhere and can be interpreted as consumption rates. The discontinuities of x are the gulps of consumption. Letting X be the linear span of the space of these consumption patterns, H&K introduce a family of norm topologies on X so that, among other things, consumptions

[1] Note, however, that one might expect equilibrium prices for consumption to fluctuate wildly in an economy under uncertainty due to temporal resolution of uncertainty; see Huang (1985a, 1985b).

[2] Throughout this paper we will use weak relations: increasing means nondecreasing, positive means nonnegative, etc. When the relations are strict, we will say, for example, strictly increasing.

at nearby dates are almost perfect substitutes. An example of this family of norm topologies is the L^p topology on cumulative consumption, where the L^p-norm of a consumption pattern x is given by:[3]

$$\left(\int_0^1 |x(t)|^p dt + |x(1)|^p\right)^{\frac{1}{p}}.$$

H&K also show that preferences continuous in any of the norm topologies can be represented by a time-additive functional of consumption rates in the "standard answers" only if the felicity functions are linear in the consumption rates. The equilibrium prices basically come from absolutely continuous functions – time zero prices of consumptions delivered at nearby future dates are almost equal and change across these dates in an almost differentiable manner.

The purpose of this paper is to develop topologies on the space of consumption patterns under uncertainty that capture the notion that consumptions at nearby dates are almost perfect substitutes. A crucial distinction between an economy under certainty and an economy under uncertainty is that in the latter the passage of time also reveals partially the true state of nature, or reveals *information.* In a world of uncertainty, the preferences of an individual over consumption patterns may be affected by the way uncertainty is resolved over time. A world in which any uncertainty is resolved gradually over time, with ample "warning" and "preparation" for new bits of information is certainly different from a world which is "sudden" and "full of surprises". Thus it is unreasonable to require *a priori* that consumptions at nearby dates, which may be random, be almost perfect substitutes unless there are no "surprises" there. Similarly, we would not expect the prices for consumption delivered at different dates in a state of nature to be a continuous function of time except at times of no surprise.

We introduce a family of norm topologies on the linear span of the set of consumption patterns under uncertainty, which is composed of processes with positive and increasing paths. The topologies are natural generalizations of the H&K topologies. In particular, they reduce to the H&K topologies when the true state of nature is revealed at time 0 and imply that consumptions at nearby dates, where there is no discontinuity of information, are almost perfect substitutes. We then show that preferences continuous in any one of the family of topologies exhibit intuitively appealing economic properties.

In general equilibrium theory and in the theory of asset valuation by arbitrage, prices normally come from the topological dual spaces. Thus we also characterize the topological dual

[3] We include the last term $x(1)$ in the definition of the norm to distinguish between two consumption patterns that are equal over the interval $[0, 1)$, but differ in that one pattern involves a "gulp" at $t = 1$, which will not be accounted for by the integral $\int_0^1 |x(t)|^p dt$.

spaces of the suggested family of norm topologies and show that they are essentially composed of processes which are sums of processes of absolutely continuous paths and martingales. This is a very natural result. H&K have shown that in the case of certainty the time zero prices for consumption delivered at different dates are essentially absolutely continuous functions of time. Hence preferences that treat consumption at nearby dates as almost perfect substitutes will give rise to nearly equal prices for consumption at nearby dates and these prices vary in an almost differentiable fashion; an intuitively attractive conclusion. In the case of uncertainty a new element, the pattern of information flow, affects the sample path properties of equilibrium prices. This effect is captured in the martingale component of the price process. It is known that a martingale can make discontinuous changes only at surprises and can fluctuate in a nowhere differentiable fashion. Thus prices for consumption delivered at different dates in a state of nature are a continuous function of time except possibly at surprises and can fluctuate in a nowhere differentiable manner (due to temporal resolution of uncertainty). This agrees with our economic intuition.

We also investigate the properties of arbitrage-free price processes for long-lived securities in a dynamic securities market economy where an arbitrage opportunity is defined using concepts of continuity derived from our family of topologies. It is shown that, between lump-sum ex-dividend dates, price processes for these securities are continuous except possibly at surprises. In particular, if the information structure is generated by a Brownian motion, then (ex-dividend) price processes plus cumulative dividends are Itô processes.

It is worthwhile to point out that earlier models of general equilibrium in asset markets using the standard representation of utility had to rely on exogenous factors in addition to preferences to characterize the sample path properties of equilibrium prices; see Duffie and Zame (1989) and Huang (1987). For example, in the case of a Brownian motion filtration, the price process plus the cumulative dividends of a security will not be an Itô process unless the aggregate endowment process is. In our setup, however, price processes of long-lived securities plus their cumulative dividends will be Itô processes independently of the properties of the aggregate endowment process.

As for the existence of an Arrow-Debreu equilibrium in an economy populated with agents whose preferences are continuous in one of our topologies, we have little to report. Known sufficient conditions for the existence of an equilibrium are not satisfied by our topologies. This opens up a question about the existence of an equilibrium in economies of our type. However, there always exists an equilibrium in an economy with a representative agent whose

preferences are continuous in one of our topologies and are uniformly proper (to be defined). The equilibrium prices lie in the topological dual space.

The rest of this paper is organized as follows. Section 2 formulates the stochastic environment under study with a time span $[0,1]$. A consumption pattern $x = \{x(t); t \in [0,1]\}$ is a stochastic process, consistent with the information flow, having positive, increasing, and right-continuous sample paths. The random variable $x(t)$ denotes the cumulative consumption from time 0 to time t. Let $\mathbf{X}_+$ be the space of consumption patterns and $\mathbf{X}$ be the linear span of $\mathbf{X}_+$. We define the commodity space and consumption set and introduce a family of topologies. Our construction of the the topologies is motivated by two considerations.

First, an economy under certainty is a special case of an economy under uncertainty where the true state of nature is revealed at time 0. In such event, we would like our topology to agree with the H&K topology. This necessitates that the norms of H&K be used path by path (state by state) on a consumption pattern. For the L^p example mentioned above, the path-wise distance between $x \in \mathbf{X}_+$ and 0 is

$$\left(\int_0^1 |x(\omega,t)|^{p(\omega)}dt + |x(\omega,1)|^{p(\omega)}\right)^{\frac{1}{p(\omega)}}, \tag{1}$$

where $x(\omega,t)$ is the value of the random variable $x(t)$ when the state of nature is ω. Note that in (1) p is a function of the state of nature ω – there is no *a priori* reason to expect that the trade-off of consumption across time is the same for all states.

Second, we consider substitutability of consumption across states. One unit of consumption at a time in a particular state of nature may be a close substitute to one unit of consumption at the same time in another state of nature for an individual. But there is no economic reason to expect that *all* individuals with continuous preferences consider consumptions in different states to be close substitutes. Thus the topology on $\mathbf{X}$ should not *a priori* build in substitutability of consumption across states in an arbitrary manner. On the other hand, it is quite intuitive that one unit of consumption in a collection of states which is very unlikely to occur should be close to not consuming at all.

With these considerations in mind, a natural way of aggregating the path-wise construction in the previous paragraph is to just take expectation. Taking expectations embodies the notion that consumptions at two disjoint events are perfect nonsubstitutes except when both events are negligible in probability and thus the two consumption patterns are almost indistinguishable from zero. Taking expectations also embodies the notion that, at any time, the differences of preferences for consumption in two events with the same probability is a decreasing function in

the degree of overlap of the two events. We here remind the reader that defining such a strong topology and considering continuous preferences certainly does not rule out preferences that actually consider consumptions across states to be close substitutes. They will be special cases of preferences continuous in the "strong" topology. In the same L^p example above, except now that p is nonrandom, the norm of a consumption pattern $x \in \mathbf{X}_+$ is then

$$\left(\mathbf{E}\left[\int_0^1 |x(t)|^p dt + |x(1)|^p\right]\right)^{\frac{1}{p}}, \tag{2}$$

which is a standard L^p norm but on cumulative consumptions rather than on consumption rates.[4]

In section 3, we show that, in fact, preferences continuous in any one of our topologies take an economically appealing view of intertemporal consumption. In particular, consumptions at nearby dates where there is no discontinuity of information are almost perfect substitutes. Section 4 characterizes the topological duals of the family of topologies under consideration.

In section 5, we examine how standard models with time-additive utility functions fare in our set up. Similar to the results of H&K, preferences represented by time-additive utility functions over consumption rates are continuous in any of our topologies if and only if they are linear. In addition, we examine some of the prevalent representations in the literature of "non time-additive" utility functions such as those in Bergman (1985), Constantinides (1988), Heaton (1990), and Sundaresan (1989). Although such representations have elements that capture the effect of past consumption on current utility, we find that most of them imply preferences that are *not* continuous in the sense that we advocate except for some functional forms studied in Heaton (1990).

In section 6 we study the structural properties of our commodity space and show that it does not satisfy the known sufficient conditions for existence of an Arrow-Debreu equilibrium. We view this as an invitation for developing more powerful existence theorems. Section 7 examines the implications of our topologies on the prices of long-lived securities in dynamic asset markets. Section 8 contains concluding remarks and Appendix A contains proofs.

2 Formulation

Consider an agent who lives in a world of uncertainty from time 0 to time 1. There is a single consumption good available for consumption at any time $t \in [0, 1]$. Uncertainty is modeled by

[4]The reader may have noticed that (2) is not finite for every $x \in \mathbf{X}_+$. Thus we will have to restrict our attention to a subset of $\mathbf{X}_+$ depending on the topology.

a complete probability space $(\Omega, \mathcal{F}, P)$. Each $\omega \in \Omega$ is a state of nature which is a complete description of one possible realization of all exogenous sources of uncertainty from time 0 to time 1. The sigma-field $\mathcal{F}$ is the collection of events which are distinguishable at time 1 and P is a probability measure on $(\Omega, \mathcal{F})$.

We take as given the time evolution of the agent's knowledge about the states of nature. This is modeled by a *filtration* $\mathbf{F} = \{\mathcal{F}_t; t \in [0,1]\}$, which is an increasing family of sub sigma-fields of $\mathcal{F}$; that is, $\mathcal{F}_s \subseteq \mathcal{F}_t$ if $s \leq t$. Interpret $\mathcal{F}_t$ as the information that the agent has at time t. We assume that $\mathcal{F} = \mathcal{F}_1$, that is, the true state of nature will be known at time 1, and the filtration $\mathbf{F}$ satisfies the following *usual* conditions:

1. $\mathbf{F}$ is *complete* in that $\mathcal{F}_0$ contains all P-null sets;

2. $\mathbf{F}$ is right continuous in that $\mathcal{F}_t = \mathcal{F}_{t+}$, where $\mathcal{F}_{t+} = \bigcap_{s>t} \mathcal{F}_s$.

To analyze preferences and prices we need to characterize the way information is revealed. For this purpose we rely on the concepts of optional and predictable times which are defined as follows:

Definition 1 *The function $T : \Omega \to [0, \infty]$ is an optional time with respect to* $\mathbf{F}$ *if*

$$\{\omega \in \Omega : T(\omega) \leq t\} \in \mathcal{F}_t \quad \forall t \in [0,1].$$

An optional time can always be interpreted to be the first time a certain event happens. The condition $\{\omega \in \Omega : T(\omega) \leq t\} \in \mathcal{F}_t$ in the above definition then says that at any time t, it will be known whether a certain event has happened. By the definition, any deterministic time t is an optional time. It is easily seen that if T is an optional time, then $T + a$ is an optional time for all $a > 0$. On the other hand, $T - a$ may not be an optional time.

Definition 2 *An optional time T is said to be predictable if there exists a sequence of optional times $\{T_n\}$ such that $T_n \leq T$ a.s. and on the set $\{T > 0\}$, almost surely, $T_n < T_{n+1} < T$ and $T_n \nearrow T$. The sequence $\{T_n\}$ is said to announce T.*

Note that for any deterministic time t we can choose $\max(0, t - \frac{1}{n})$ to be its announcing sequence. Thus any deterministic time t is predictable.

An important characteristic of an information structure is whether it has sudden changes in the information content at known or predictable times. An information structure with the property that all the information at a predictable optional time is the same as all the information an instant before is said to be *quasi left-continuous*. The following is the precise definition.

Definition 3 *An information structure* $\mathbf{F}$ *is quasi left-continuous if for a predictable time* T *and its announcing sequence* $\{T_n\}$ *we have*

$$\mathcal{F}_T = \mathcal{F}_{T-} \equiv \bigvee_{n=1}^{\infty} \mathcal{F}_{T_n},$$

where $\mathcal{F}_T$ *is a sigma-field of events prior to* T *consisting of all events* $A \in \mathcal{F}$ *such that*

$$A \bigcap \{T \leq t\} \in \mathcal{F}_t, \text{ for all } t \in [0,1].$$

The filtration generated by a diffusion process, a Poisson process, or a combination of the two is quasi left-continuous once the filtration is completed. Moreover, it has been shown by Meyer (1963) that any information structure generated by a process that is continuous at predictable optional times (defined with respect to its natural filtration) and has the strong Markov property is quasi left-continuous.

An information structure which is not quasi left-continuous arises in situations in which the time of an event is known but the details of the event are not revealed or predicted before that time. For example, announcements of major policy changes when the date of an announcement is known beforehand, while the actual content of the announcement is only revealed on that date.

We assume throughout our analysis that $\mathbf{F}$ is quasi left-continuous. This assumption is not essential for our results. However, it simplifies our intuitive interpretation of the results as then an anticipated event is synonymous to a predictable optional time. In the concluding remarks, we will detail how our results can be generalized to economies with an information structure that is not quasi left-continuous.

A stochastic process Y is a mapping $Y : \Omega \times [0,1] \to \Re$ that is measurable with respect to $\mathcal{F} \otimes \mathcal{B}([0,1])$, the product sigma-field generated by $\mathcal{F}$ and the Borel sigma-field of $[0,1]$. For each $\omega \in \Omega$, $Y(\omega, \cdot) : [0,1] \to \Re$ is a *sample path* and for each $t \in [0,1]$, $Y(\cdot, t) : \Omega \to \Re$ is a random variable, which we will usually denote by $Y(t)$. The process Y is said to be adapted to $\mathbf{F}$ if for each $t \in [0,1]$, $Y(t)$ is $\mathcal{F}_t$-measurable. This is a natural information constraint: the value of the process at time t cannot depend on the information yet to be revealed in the future. For brevity, all the processes to appear will be measurable and adapted to $\mathbf{F}$ unless otherwise mentioned.

We represent the life time consumption pattern of the agent by a process x whose sample paths are positive, increasing, and right-continuous where $x(\omega, t)$ denotes the cumulative consumption from time 0 to time t in state ω. The set of such processes is $\mathbf{X}_+$. The linear span of

$\mathbf{X}_+$, the space of processes having right-continuous and bounded variation sample paths, will be denoted by $\mathbf{X}$. We restrict our attention to the consumption set and the commodity space, which are subsets of $\mathbf{X}_+$ and $\mathbf{X}$, respectively, defined as follows.

Let Φ be the collection of functions $\varphi : \Omega \times \Re_+ \to \Re_+$ measurable with respect to $\mathcal{F} \otimes \mathcal{B}(\Re)$ with, *almost surely*,

$$\varphi(\omega, 0) = 0 \quad \text{and} \quad \varphi(\omega, \infty) = \infty,$$

which are continuous, strictly increasing, convex, and integrable in that

$$\int_\Omega \varphi(\omega, z) P(d\omega) < \infty \quad \forall z > 0.$$

For a $\varphi \in \Phi$ let $\mathcal{E}^\varphi$ be the vector subspace of $\mathbf{X}$ defined as:

$$\mathcal{E}^\varphi = \left\{ x \in \mathbf{X} \colon E\left[\int_0^1 \varphi\Big(\gamma |x(t)|\Big)\, dt + \varphi\Big(\gamma |x(1)|\Big) \right] < \infty \quad \forall \gamma > 0 \right\}, \tag{3}$$

and define the following norm on $\mathcal{E}^\varphi$:

$$\|x\|_\varphi = \inf\left\{ a > 0 : E\left[\int_0^1 \varphi(\frac{|x(t)|}{a})\, dt + \varphi(\frac{|x(1)|}{a}) \right] \leq 1 \right\}. \tag{4}$$

(For the fact that (4) defines a norm on $\mathcal{E}^\varphi$, see Musielak (1983, theorem 1.5).) For simplicity, we will use $\mathcal{T}_\varphi$ to denote the topology on $\mathcal{E}^\varphi$ generated by $\|\cdot\|_\varphi$.

The class of topologies generated by elements of Φ is a natural and simple generalization of the familiar L^p topologies defined on cumulative consumption. In contrast, existing dynamic models of asset prices such as Duffie and Zame (1989) and Huang (1987) use L^p topologies defined on consumption "rates", which are essentially equivalent to the total variation topology on cumulative consumption and are much stronger topologies.

Henceforth we will use $\mathcal{E}^\varphi$ for some $\varphi \in \Phi$ to be the commodity space of the economy that we study. The agent in the economy has a preference relation $\succeq$ defined on $\mathcal{E}^\varphi_+$, where $\mathcal{E}^\varphi_+$ denotes the positive orthant of $\mathcal{E}^\varphi$ and is the agent's consumption set. We assume that $\succeq$ is continuous with respect to $\mathcal{T}_\varphi$. We will demonstrate in the analysis to follow that $\succeq$ exhibits economically appealing properties.

3 Properties of Continuous Preferences

In this section we will show that the topology $\mathcal{T}_\varphi$ gives economically reasonable sense of "closeness" for consumption patterns in $\mathcal{E}^\varphi_+$. In particular, $\succeq$ treats consumptions at nearby (possibly

random) dates where there are no unforeseen events as almost perfect substitutes. In addition, $\succeq$ regards consumption patterns that differ slightly in the quantity and the timing of consumption as close commodities. We will accomplish these by establishing the convergence of several sequences in three propositions. Continuity of preferences should be understood in economic terms as follows: Let $x_n, x, y \in \mathcal{E}_+^\varphi$ and $x_n \to x$ in $\mathcal{T}_\varphi$ as $n \to \infty$. Then, $x_n \succeq y$ for all n implies $x \succeq y$ and $y \succeq x_n$ for all n implies $y \succeq x$.

First, we show, roughly, that two patterns of consumption which have almost equal cumulative consumption at every point in time and in each state are close.

Proposition 1 *Let $x, x_n \in \mathcal{E}_+^\varphi$ for all n. If $\lim_{n\to\infty} \sup_{t\in[0,1]} |x_n(\omega, t) - x(\omega, t)| = 0$ a.s. and if there exists a random variable K with the property that $\mathrm{E}[\varphi(K)] < \infty$, such that $\sup_{t\in[0,1]} |x_n(\omega, t) - x(\omega, t)| \leq K(\omega)$ a.s., then $x_n \to x$ in $\mathcal{T}_\varphi$ as $n \to \infty$.*

Second, we show that a delay or an advancement of a sizable quantity of consumption within the "information constraint" for a small enough time is regarded as insignificant. The information constraint is crucial here. One unit of consumption at time t in an event $A \in \mathcal{F}_t$ may not be advanced to time $t - \epsilon$ in the same event for $\epsilon > 0$ however small, since A may not be a distinguishable event in $\mathcal{F}_{t-\epsilon}$ and doing so violates the information constraint that consumption patterns be adapted to $\mathbf{F}$. On the other hand, one unit of consumption at time t in an event can always be delayed to any time $s > t$ in the same event. Moreover, one unit of consumption at an unforeseen event cannot be advanced to an instant before by the nature of a surprise.

The following proposition formalizes this discussion.

Proposition 2 *Let T be an optional time with $P\{T = 1\} = 0$ and k be a strictly positive scalar. Then*

$$\|k\chi_{[T+\frac{1}{n},1]} - k\chi_{[T,1]}\|_\varphi \to 0 \text{ as } n \to \infty,$$

where χ is an indicator function. On the other hand, suppose that $\{T_n\}$ is a sequence of optional times with $T_n \leq T$, and on the set $\{T > 0\}$, $T_n < T_{n+1} < T$. Then

$$\|k\chi_{[T_n,1]} - k\chi_{[T,1]}\|_\varphi \to 0 \text{ as } n \to \infty$$

only if T is predictable. Conversely, if T is a predictable optional time with an announcing sequence $\{T_n\}$, then

$$\|k\chi_{[T_n,1]} - k\chi_{[T,1]}\|_\varphi \to 0 \text{ as } n \to \infty.$$

Note that since any deterministic t is predictable, the second assertion of this proposition shows that one unit of sure consumption at time t is a close substitute of one unit of sure consumption at $t - \frac{1}{n}$ when n is large. This together with the first assertion of this proposition implies that sure consumptions at nearby dates are close substitutes.

We conclude our investigation of the flexibility provided by the topology $\mathcal{T}_\varphi$ in allowing substitution of consumption between adjacent times by analyzing a rather more general trade-off between the quantity and the times of consumption. In Proposition 2, we showed that delaying or advancing a *fixed* amount of consumption for a small enough period results in a very small change in total satisfaction. In the following analysis, consumption patterns differ slightly not only in the timing of consumption but also in the amounts consumed.

Let $x, y \in \mathcal{E}_+^\varphi$ and define a Prohorov metric path-by-path:

$$\rho_\omega(x, y) = \inf\{\epsilon \geq 0: x(t+\epsilon, \omega) + \epsilon \geq y(t, \omega) \geq x(t-\epsilon, \omega) - \epsilon, \forall t \in [0,1]\}.$$

If $y(\omega, \cdot)$ is within ϵ of $x(\omega, \cdot)$ in the Prohorov sense, then at every time t in state ω, the total consumption under x is within ϵ of the total consumption under y at some time no more than ϵ away from t.

The idea is that if $\rho_\omega(x, y)$ is uniformly smaller than ϵ across states of nature, then sizable shifts of consumption between x and y at predictable optional times can only occur in a time interval less than ϵ in width. Such shifts should not drastically change an agent's utility if x were exchanged for y. At non-predictable times, however, Proposition 2 implies that as $\epsilon \to 0$, the ϵ neighborhood of x does not contain any y that advances consumption to earlier time since that violates the information constraint. As it turns out, $\rho_\omega(x, y)$ being uniformly small is too strong, and we can allow $\rho_\omega(x, y)$ to be large on a set of small probability.

Proposition 3 *Let $x, x_n \in \mathcal{E}_+^\varphi$ with the property that there exists an $\mathcal{F}$-measurable function, say K, such that*

$$\rho_\omega(x_n, x) \leq \frac{K(\omega)}{n} \quad P\text{-a.s.} \quad \text{and} \quad \mathbf{E}[\varphi(K)] < \infty.$$

Then $\|x_n - x\|_\varphi \to 0$ as $n \to \infty$.

4 Duality

The preceding analysis shows that the agent treats consumptions at nearby (possibly random) dates as very similar "goods" except possibly at information surprises. From economics, one

expects almost perfect substitutes to have almost equal prices. This intuition is confirmed by the results of this section.

In standard general equilibrium theory of the Arrow-Debreu sort and in the theory of asset valuation by arbitrage, prices come from the space of linear functionals continuous in the topology with respect to which agents' preferences are assumed to be continuous; for the former see, e.g., Bewley (1972) and Mas-Colell (1986), and for the later see Kreps (1981). This space of continuous linear functionals is called the *topological dual spacc.*[5] In particular, Mas-Colell (1986) showed that there always exists an equilibrium in a representative agent pure exchange economy with prices from the topological dual if the agent's preferences satisfy a strong monotonicity condition. For our analysis here, we will assume that prices indeed come from the topological dual space and characterize this space.

Let ψ be a continuous linear functional that gives either equilibrium or arbitrage-free prices for consumption patterns. Then $\psi(x)$ is the price at time 0 of the consumption claim x. We will find out soon that ψ can be represented in the form

$$\psi(x) = \mathbf{E}\left[\int_{0-}^{1} g(t)dx(t)\right] \quad x \in \mathcal{E}^{\varphi},$$

where, roughly, g is a process that is decomposable into two parts: a process having absolutely continuous sample path and a martingale.[6] Note that $g(\omega, t)$ is the time 0 price of one unit of consumption at time t in state ω, per unit of probability. For simplicity, we will henceforth refer to $g(t)$ as the "shadow price for time t consumption." Since the sample path properties of a martingale are solely determined by how information about the state of nature is revealed over time, we can then characterize sample path properties of g from studying the properties of the information structure $\mathbf{F}$.

We now proceed to characterize the topological dual of $(\mathcal{E}^{\varphi}, \mathcal{T}_{\varphi})$, denoted by $\mathcal{E}^{\varphi^*}$. Some definitions are in order.

Definition 4 *A function $\varphi \in \Phi$ is said to be an N-function if*

$$\lim_{x \downarrow 0} \frac{\varphi(\omega, x)}{x} = 0 \quad \text{and} \quad \lim_{x \uparrow \infty} \frac{\varphi(\omega, x)}{x} = \infty \quad \text{for all} \quad \omega \in \Omega\,.$$

By convexity, for φ an N-function, we can write $\varphi(\omega, |x|) = \int_0^{|x|} v(\omega, s)\, ds$, where $v(\omega, s)$ is

[5] More formally, $(\mathcal{E}^{\varphi}, \mathcal{T}_{\varphi})$ is a topological linear space and the space of continuous linear functionals on $(\mathcal{E}^{\varphi}, \mathcal{T}_{\varphi})$ is its topological dual.

[6] A martingale Y is a process (adapted to $\mathbf{F}$) that has right-continuous sample paths with the property that $\mathrm{E}[Y(s)|\mathcal{F}_t] = Y(t)$ *a.s.* for all $s \geq t$.

the right-hand derivative of $\varphi(\omega, s)$ for a fixed ω. Let $v^*(\omega, \sigma) = \sup\{s: v(\omega, s) \leq \sigma\}$, then $\varphi^*(\omega, x) \equiv \int_0^{|x|} v^*(\omega, \sigma)\, d\sigma$ is said to be complementary to φ.

Example 1 *(1) If $\varphi(\omega, x) = |x|^{p(\omega)}, 1 < p(\omega) < \infty$, then $\varphi^*(\omega, y) = |y|^{q(\omega)}$, where $\frac{1}{p(\omega)} + \frac{1}{q(\omega)} =$ 1. (2) If $\varphi(\omega, x) = e^x - x - 1$, then $\varphi(\omega, y) = (1+y)ln(1+y) - y$.*

The following proposition describes $\mathcal{T}_\varphi$-continuous linear functionals on the space $\mathcal{E}^\varphi$.

Proposition 4 *Let $\varphi \in \Phi$ be an N-function and with a complementary function φ^*. Then $\psi : \mathcal{E}^\varphi \to \Re$ is a $\mathcal{T}_\varphi$-continuous linear functional if and only if there exists an adapted process f with absolutely continuous sample paths, and with*

$$f' \in L^{\varphi^*} \equiv \left\{x \in \mathbf{O}: E\left[\int_0^1 \varphi^*(\gamma|x(t)|)\, dt + \varphi(\gamma|x(1)|)\right] \to 0 \quad as \quad \gamma \downarrow 0\right\}.$$

so that

$$\psi(x) = \mathbf{E}\left[\int_{0-}^1 g(t) dx(t)\right] \quad \forall x \in \mathcal{E}^\varphi, \tag{5}$$

$$where \quad g(t) = f(t) + m(t), \qquad and \tag{6}$$

$$m(t) = \mathbf{E}[-f'(1) - f(1)|\mathcal{F}_t] \quad \forall t \in [0, 1] \tag{7}$$

is a martingale, or equivalently, $g(t) = f(t) - f'(1) - f(1)$, which is a process with absolutely continuous paths but may not be adapted.

An important special case of Proposition 4 is when $\varphi(\omega, z) = |z|^p$ with $1 < p < \infty$. The case when $p = 1$ is not covered in Proposition 4, however, since $|z|$ is not an N function. The following proposition, whose proof is left for the reader, gives the duality result for this L^p family.

Proposition 5 *Let $\varphi(\omega, z) = |z|^p$, for all ω, with $1 \leq p < \infty$. Let q be such that $1/p + 1/q = 1$. Then $\psi : \mathcal{E}^\varphi \to \Re$ is a $\mathcal{T}_\varphi$-continuous linear functional if and only if there exists an adapted process f with absolutely continuous sample paths, with*

$$(f', f'(1)) \in L^q(\Omega \times [0, 1], \mathcal{O}, P \times \lambda) \times L^q(\Omega, \mathcal{F}, P),$$

so that

$$\psi(x) = \mathbf{E}\left[\int_{0-}^1 g(t) dx(t)\right] \quad \forall x \in \mathcal{E}^\varphi, \tag{8}$$

$$\textit{where} \quad g(t) = f(t) + m(t), \quad \textit{and} \tag{9}$$

$$m(t) = \mathrm{E}[-f'(1) - f(1)|\mathcal{F}_t] \quad \forall t \in [0,1] \tag{10}$$

is a martingale, or equivalently, $g(t) = f(t) - f'(1) - f(1)$, *which is a process with absolutely continuous paths but is not necessarily adapted.*

Note that from the above proposition, when $p = 1$, the dual space is composed of processes that are sums of processes with Lipschitz continuous paths and bounded martingales. Another case of the function φ which is not covered in the above propositions is when φ is asymptotically linear for some ω's, in that we have $\lim_{x\to\infty} \frac{\varphi(\omega,x)}{x} = \alpha > 0$ for some $\omega \in \Omega$. In the case of certainty, H&K have shown that the topological dual in this case is the space of Lipschitz continuous functions. We obtain a similar characterization for the case of uncertainty under the assumption that the function φ is "integrably asymptotically linear." This result is recorded in the following proposition.

Proposition 6 *Let the function* φ *be integrably asymptotically linear, in that for any* $\epsilon > 0$, *there exists a scalar* $\alpha > 0$ *and a random variable* K, *with the property that* $\mathrm{E}[\varphi(K)] < \infty$, *such that*

$$\varphi(\omega, x) \leq \alpha x + \epsilon \quad \forall x \geq K(\omega) \ a.s. \tag{11}$$

Then $\mathcal{E}^\varphi$ *and* $\mathcal{T}_\varphi$ *are the same as the consumption space and the topology constructed using* $\varphi(\omega, x) = |x|$ *for all* $\omega \in \Omega$. *Moreover, an element of the topological dual space of* $\mathcal{E}^\varphi$ *is the sum of a process with Lipschitz continuous sample paths and a bounded martingale.*

In the case of certainty, a continuous linear functional can be represented essentially by an absolutely continuous function (see H&K or note that under certainty, a martingale must be a constant). Here, however, a continuous linear functional is represented by a process that is the sum of an absolutely continuous process and a martingale. It is known that the sample path properties of a martingale are determined by the way information is revealed over time: A martingale can have a discontinuity only at non-predictable optional times; see, for example, Meyer (1966, Theorem VI.14). It then follows that the shadow prices for consumption at nearby (possibly random) dates are almost equal when there are no "surprises."

An information structure $\mathbf{F}$ is said to be continuous if $P(A|\mathcal{F}_t) = \mathrm{E}[\chi_A|\mathcal{F}_t]\ a.s.$ is a process with almost all its sample paths continuous for all $A \in \mathcal{F}$; that is, the posterior probability of any event evolves continuously over time. It is known that an information structure is

continuous if and only if all optional times are predictable; see Huang (1985a, theorem 6.3). Thus when **F** is continuous, $\mathcal{E}^{\varphi^*}$ consists only of continuous processes and thus the shadow prices of consumption at nearby (random) dates are almost equal. It is also known that if a martingale is continuous on a stochastic interval[7], it is either a constant or of unbounded variation there (Fisk (1965)). Hence there are cases where the shadow prices for consumption at nearby dates are almost equal but fluctuate wildly in a nowhere differentiable fashion. This happens because of the temporal resolution of uncertainty. When there is no uncertainty, the martingale part of the shadow prices for consumption becomes a constant and these prices degenerate to absolutely continuous functions of time.

An important special case of continuous information structure is when **F** is generated by a Brownian motion. In this case, the dual space contains Itô processes only. This characterization is given in the following proposition:

Proposition 7 *Let* **F** *be generated by a Brownian motion and let g be an element of any of the dual spaces characterized in Propositions 4, 5 and 6, then g is an Itô process:*

$$g(t) = \int_0^t f'(s)\,ds + \int_0^t \theta(s)^\top\,dw(s),$$

where f is a process with absolutely continuous paths and f' denotes its derivative with respect to time, where w is a Brownian motion and where θ is adapted to **F** *and satisfies:*

$$\int_0^1 |\theta(t)|^2\,dt < \infty \quad a.s.$$

Note that in the existing dynamic models of asset prices in pure exchange economies in continuous time such as Duffie and Zame (1989) and Huang (1987), agents' learn the true state of nature by observing a Brownian motion and the shadow prices for consumption are represented by an Itô process. There agents' preferences are defined only over consumption rates and are continuous with respect to the L^p topology on the rates. Since these L^p topologies on consumption rates are much stronger topologies than $\mathcal{T}_\varphi$, their topological duals contain processes that are not Itô processes. Thus Duffie and Zame and Huang had to assumed that agents' preferences were time-additive and that the aggregate endowment process was an Itô process. Here, however, we have a much weaker topology that treats consumption at nearby dates as close substitutes (except possibly at information surprises). Hence shadow prices for consumption are Itô processes independent of the special properties of the aggregate endowment process; see, Proposition 7.

[7] Let $T \leq \tau$ a.s. be two optional times. Then $[T,S] = \{(\omega,t) : T(\omega) \leq t \leq \tau(\omega)\}$ is an stochastic interval.

Note also that an example of a discontinuous information structure is when information is obtained by observing the realizations of a simple Poisson process $\{N(t); t \in [0,1]\}$ with intensity λ.[8] In this case, a martingale can be represented by a stochastic integral with respect to the Poisson martingale, $N(t) - \lambda t$, and thus may have a discontinuity only when the Poisson process jumps; see, e.g., Davis (1973). As a consequence, the shadow price process can have a discontinuity only at a "surprise" which occurs when the Poisson process jumps.

5 On Standard Models

In this section we examine how the standard models fare in the context of our current model. Using a standard model, one assumes that an agent maximizes his expected utility of the following form:

$$U(x) = \mathbf{E}\left[\int_0^1 u(x'(t), t)dt\right], \tag{12}$$

for absolutely continuous consumption patterns x, where $u(z,t)$ is a time-additive "felicity" function defined on consumption rates and where $x'(t)$ denotes the time derivative of $x(t)$ at t. Although such a model does not permit gulps of consumption, this restriction creates no major problem. One can easily show that the set of absolutely continuous consumption patterns is dense in $\mathcal{E}^{\varphi}$. We can therefore define the expected utility of a consumption pattern involving gulps to be the limit of the expected utilities of a sequence of approximating absolutely continuous consumption patterns. This procedure works provided that $U(x)$ is $\mathcal{T}_{\varphi}$-continuous. We show, however, in what follows that if $u(z,t)$ is jointly continuous then $U(x)$ is $\mathcal{T}_{\varphi}$-continuous only if $u(z,t)$ is linear in z.

Proposition 8 *Let $u(z,t) : \Re_+ \times [0,1] \to \Re$ be jointly continuous. The utility function $U : \mathcal{E}^{\varphi}_+ \to \Re$ is continuous in $\mathcal{T}_{\varphi}$ only if there exists continuous functions $\alpha : [0,1] \to \Re$ and $\beta : [0,1] \to \Re$ such that $u(z,t) = \alpha(t)z + \beta(t)$.*

The converse of the above proposition is a direct application of the duality results in Section 4.

Proposition 9 *Let $u(z,t) = \alpha(t)z + \beta(t)$. Then U of (12) is continuous in $\mathcal{T}_{\varphi}$ if $\alpha' \in L^{\varphi^*}$, where $\alpha'(t)$ is the derivative of $\alpha(t)$.*

[8] For this process, over the time interval $[0,t]$, the probability of a jump of unit one is $1 - \exp\{-\lambda t\}$.

· The above analysis reveals that the standard representation of utility as the expectation of the integral of felicity of current consumption fails (except in a very special case) to produce preferences continuous in our topologies. The problem is that the additive form of the felicity of current consumption implies that consumption at an earlier date, no matter how recent it is, has no effect on current satisfaction. This is in direct conflict with our concept of preference continuity in which consumptions at nearby dates are close substitutes if there is no discontinuity of information.

The standard model also fails to capture our economic intuition in situations different from the single perishable consumption good case that we deal with here. Take for example the case of durable goods. One could interpret the consumption plan $x(t)$ as the level of the cumulative purchases of durable good up to time t. A durable good is most often acquired in single units, and only a few times over the horizon $[0,1]$. However, the owner of the good receives a continuous flow of services from the earlier acquired durable goods. His level of satisfaction from owning the good at any time should reflect his enjoyment from the services provided by the good, despite of the fact that the good was purchased in the past.

We propose an alternative representation of the utility, which keeps the spirit of the standard model and gives rise to preferences continuous in our topologies. To achieve this objective, we express the utility as an integral of felicity as in the standard model. However, the felicity at time t depends only on a weighted average of the consumption in the "recent past." Our construction goes as follows: let $x(\omega, t)$ be a consumption plan. Consider an adapted process $\theta(\omega, t)$, which is uniformly bounded across the states of nature ω, which is differentiable in t, and whose first derivatives are uniformly bounded over ω. An example of such a process is the Gauss-kernel given by:

$$\theta(\omega, t) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-t^2}{2\sigma^2}}. \tag{13}$$

Now using the process θ, construct from x another process $\hat{x}$ which gives a path-by-path weighted average of recent past consumption. In other words, put

$$\hat{x}(\omega, t) = \int_0^k \theta(\omega, s)\, dx(t-s). \tag{14}$$

The new process $\hat{x}$ is the element over which preferences are expressed using the standard model. Let $u(\hat{x}, t)$ be a felicity function which is state-independent, continuous and concave in $\hat{x}$. Let

$$U(x) = \mathbf{E}[\int_0^1 u(\hat{x}, t)\, dt]. \tag{15}$$

The fact that under these conditions, the utility function given in (15) is continuous in any of our topologies is recorded in the following proposition.

Proposition 10 *Let the averaging process θ and the felicity function u satisfy the conditions described above. Let $U(x)$ be constructed as in (15). Preferences represented by U are continuous in any $\mathcal{T}_\varphi$.*

It is important to point out that many authors have attempted to capture the effects of substitutability of consumption across time using functional forms that appear to be similar to (15). It turns out, however, that there is a crucial difference. In our proposed representation of utilities, the felicity function u depends only on the "smoothed" consumption process $\hat{x}$ and time. In contrast, most "non time-additive" formulations in the literature posit a felicity function, say v, which takes as arguments the current consumption "rate" $x'(t)$ together with the "smoothed" earlier consumption $\hat{x}(t)$. Thus one represents preferences by:

$$V(x) = \mathbf{E}[\int_0^1 v(x'(t), \hat{x}(t), t)\, dt]. \tag{16}$$

Examples of such representations include Bergman (1985), Constantinides (1988), Heaton (1990), and Sundaresan (1989). Inclusion of the "smoothed" recent consumption in the instantaneous felicity function v captures the effect of past consumption on one's current satisfaction. However, including current consumption in the felicity function destroys the continuity of preferences in the sense we argue for in this paper except for certain special cases. We record this fact in the following proposition.

Proposition 11 *Let $v(c, z, t)\colon \Re_+^2 \times [0,1]$ be jointly continuous. The utility function $V\colon \mathcal{E}_+^\varphi \to \Re$ defined in (16) is continuous in $\mathcal{T}_\varphi$ only if there exists jointly continuous functions $\alpha\colon \Re_+ \times [0,1] \to \Re$ and $\beta\colon \Re_+ \times [0,1] \to \Re$, and a subset Λ of $\Re_+$ with a strictly positive Lebesgue measure such that $v(c, z, t) = \alpha(z, t)c + \beta(z, t)$ for all $z \in \Lambda$.*

An immediate consequence of Proposition 11 is that any felicity function $v(c, z, t)$ that is strictly concave in c cannot represent preferences continuous in $\mathcal{T}_\varphi$. For example, in the habit formation models of Constantinides (1988) and Sundaresan (1989), the felicity functions are strictly concave functions of $c - z$ and thus the preferences they represent do not treat consumption at nearby dates as close substitutes.

6 Existence of Equilibrium

In this section, we report some results relating to the existence of an Arrow-Debreu equilibrium in a pure exchange economy with a commodity space $\mathcal{E}^{\varphi}$ equipped with the topology $\mathcal{T}_{\varphi}$.

First, as mentioned before, there always exists an equilibrium with equilibrium prices in $\mathcal{E}^{\varphi^*}$ when there exists a representative agent whose preferences are continuous in $\mathcal{T}_{\varphi}$, convex, monotone, and uniformly proper with respect to $\mathcal{T}_{\varphi}$ in the direction of the aggregate endowment $e \neq 0$;[9] see Mas-Colell (1986).

Note that uniformly proper preferences have appealing economic implications. Such preferences regard as negligible changes in consumption in which increasingly larger amounts of consumption are delayed or advanced for a decreasingly smaller periods of time. The interested reader can consult H&K (proposition 5) and Hindy and Huang (1989, propositions 8 and 9) for details.

Now we turn our attention to economies with heterogeneous agents. Some definitions are in order. The consumption set $\mathcal{E}^{\varphi}_{+}$, which is the positive orthant of $\mathcal{E}^{\varphi}$, is a cone with an empty interior. The cone $\mathcal{E}^{\varphi}_{+}$ defines an order on $\mathcal{E}^{\varphi}$ in the following way: Let $x, y \in \mathcal{E}^{\varphi}$. We say x is "greater" than y, denoted by $x \geq y$, if $x - y \in \mathcal{E}^{\varphi}_{+}$. Using this notion of order, we can define the maximum and the minimum of two consumption patterns and endow the commodity space with a lattice structure.

The order dual of $\mathcal{E}^{\varphi}$ is composed of not necessarily adapted positive processes g so that the integral

$$\mathbf{E}\left[\int_{0-}^{1} g(t)dx(t)\right]$$

is well-defined and finite for $x \in \mathcal{E}^{\varphi}_{+}$. The order generated by the order dual is then the pointwise order; that is, $g_1 \leq g_2$ if $g_1(\omega, t) \leq g_2(\omega, t)$ a.e.

Since $(\mathcal{E}^{\varphi}_{+}, \mathcal{T}_{\varphi})$ has an empty interior, existing general equilibrium theories offer two possibilities. First, if $(\mathcal{E}^{\varphi}, \mathcal{T}_{\varphi})$ is, among other things, a topological vector lattice,[10] then an equilibrium exists provided that agents' preferences are uniformly proper in the direction of the aggregate endowment $e \neq 0$; see Mas-Colell (1986). Second, an equilibrium exists if, among other things, the topological dual of $(\mathcal{E}^{\varphi}, \mathcal{T}_{\varphi})$ is a lattice using the order defined by the order dual; see Mas-Colell and Richard (1987). Unfortunately, $(\mathcal{E}^{\varphi}, \mathcal{T}_{\varphi})$ is not a topological vector lattice as the

[9] A preference relation $\succeq$ on $\mathcal{E}^{\varphi}_{+}$ is uniformly proper (with respect to $\mathcal{T}_{\varphi}$) in the direction of $e \in \mathcal{E}^{\varphi}_{+}$ with $e \neq 0$ if there exists a $\mathcal{T}_{\varphi}$ open neighborhood O of the origin such that, for every $x \in \mathcal{E}^{\varphi}_{+}$, $o \in O$, and scalar $a > 0$, $x - ae + ao \not\succeq x$.

[10] $(\mathcal{E}^{\varphi}, \mathcal{T}_{\varphi})$ is a topological vector lattice if the minimum and maximum operations are continuous in $\mathcal{T}_{\varphi}$.

following proposition shows.

Proposition 12 *$(\mathcal{E}^{\varphi}, \mathcal{T}_{\varphi})$ is not a topological vector lattice.*

Furthermore, the following example demonstrates that the topological dual space of $(\mathcal{E}^{\varphi}, \mathcal{T}_{\varphi})$ is not a lattice in the pointwise order.

Example 2 *Let the filtration be a Brownian motion filtration. From Proposition 7 we know that the topological dual contains only Itô processes. It is well known that the pointwise minimum of two Itô processes is a generalized Itô process[11] (see, e.g., Harrison (1985), §6). That is, by taking the pointwise minimum of two Itô processes, one creates a process whose time trend part may have a singular component.[12] Thus the space of Itô processes is not itself a lattice using the pointwise order. We note, in contrast, that the space of generalized Itô processes is indeed a lattice using the pointwise order.*

The known sufficient conditions for existence of an Arrow-Debreu equilibrium are not satisfied in our model, and hence existence of equilibrium is still an open question. We hope that more powerful existence theorems capable of accommodating our model will be developed in the future.

7 Arbitrage Pricing of Financial Assets

Although we have not established existence of an Arrow-Debreu equilibrium in our model, we can still provide useful results about the price behavior of financial assets using the notion of the absence of arbitrage possibilities. Using a similar commodity space and a Mackey topology generated by the space of bounded and continuous processes and barring arbitrage opportunities, Huang (1985b) has shown that security prices processes are continuous at predictable optional times when no lump-sum dividends are distributed there. In particular, price processes are generalized Itô processes between lump-sum ex-dividend dates if the information structure is generated by a Brownian motion. However, requiring the dual of the commodity space to have only continuous processes is a bit too strong. It implies that consumption at nearby dates are almost perfect substitutes even at times of surprise. Using the family of economically more reasonable topologies we advanced, we get results slightly stronger than those of Huang (1985b) in the following manner.

[11] A generalized Itô process is a continuous process with bounded variation sample paths plus an Itô integral.

[12] A function $g(t)$ is singular if it is continuous, is not a constant, and has zero derivative almost everywhere.

Assume that the commodity space $(\mathcal{E}^{\varphi}, \mathcal{T}_{\varphi})$ is separable.[13] There are N long-lived securities[14] which are available for trading any time in $[0,1]$. Security n is denoted by its cumulative dividend process $x_n \in \mathcal{E}^{\varphi}$. Let $S_n(t)$ denote the ex-dividend price process for security n at time t. Since securities are traded ex-dividends, we assume that $x_n(0) = 0$ for all n and $S_n(1) = 0$; that is, there is no cumulative dividend at time 0 and the ex-dividend price of a security at time 1 is zero. A trading strategy θ is an N-dimensional process that prescribes the portfolio strategy for the traded securities. Without getting into technical details, we shall only allow agents to use *simple trading strategies* – strategies that are adapted, bounded, left-continuous, and change their values at a finite number of non-random time points.[15] We shall say that $m \in \mathcal{E}^{\varphi}$ is *marketed* if there is a strategy θ so that

$$\theta(t)^{\top} S(t) + m(t-) - m(0) = \theta(0)^{\top} S(0) + \int_0^t \theta(s)^{\top} dS(s) + \int_0^{t-} \theta(s)^{\top} dx(s),$$

where $^{\top}$ denotes "transpose," $x(t) = (x_1(t), \ldots, x_N(t))^{\top}$, and the integrals are defined path-by-path; alternatively, we say that m is *financed* by θ. Let M denote the space of marketed consumption patterns. It is clear that M is a linear subspace of $\mathcal{E}^{\varphi}$. An element $m \in M$ financed by θ is a *simple free lunch* if $m(0) + \theta(0)^{\top} S(0) \le 0$, $m \in \mathcal{E}^{\varphi}_{+}$, and $m \neq 0$. Barring simple free lunches, each $m \in M$ has a unique price at time 0, $m(0) + \theta(0)^{\top} S(0)$, where θ finances m, and we can define a linear functional on M by $\pi(m) = m(0) + \theta(0)^{\top} S(0)$.

Now we show that the linear functional π has a nice representation if the securities market does not admit *free lunches*, a concept due to Kreps (1981). A *free lunch* is a sequence $\{(m_n, x_n) \in M \times \mathcal{E}^{\varphi}; n = 1, 2, \ldots\}$ and a bundle $k \in \mathcal{E}^{\varphi}_{+}$ with $k \neq 0$ such that

$$m_n - x_n \in \mathcal{E}^{\varphi}_{+} \quad \forall n, \quad \|x_n - k\|_{\varphi} \to 0,$$

and

$$\liminf_n \pi(m_n) \le 0.$$

Suppose that there are no free lunches. Theorem A.1 of Duffie and Huang (1986) shows that there exists an extension of π to all of $\mathcal{E}^{\varphi}$, ψ, strictly positive in that $\psi(x) > 0$ if $x \in \mathcal{E}^{\varphi}_{+}$ and $x \neq 0$.

[13] $(\mathcal{E}^{\varphi}, \mathcal{T}_{\varphi})$ is a separable topological vector space if $(\Omega, \mathcal{F}, P)$ is separable.

[14] A security is a long-lived security if it is available for trading at all t.

[15] To consider strategies more general than simple strategies, we first need to discuss how general stochastic integrals are defined, which we do not want to do here. Interested readers should consult Huang (1985b) for a discussion in a context similar to our setup here.

Let φ satisfy conditions of Propositions 4, 5, or 6, then

$$\varphi(x) = \mathbf{E}\left[\int_{0-}^{1} g(t)dx(t)\right] \quad \forall x \in \mathcal{E}^{\varphi},$$

for some strictly positive process g, which is the sum of a martingale and a process of absolutely continuous sample paths. Let $S_m(t)$ be the ex-dividend price at time t of $m \in M$. It follows from Proposition 5.1 of Huang (1985b) that

$$S_m(t) = \frac{\mathbf{E}\left[\int_0^1 g(s)dm(s)|\mathcal{F}_t\right] - \int_0^t g(s)dm(s)}{g(t)}. \tag{17}$$

The first term of the numerator of (17) is a martingale and thus the properties of g translate into those of S_m in a natural way. For example, S_m must be continuous at predictable optional times if m is also continuous there and, between discontinuities of m, S_m can have discontinuities only at surprises. In particular, suppose that $\mathbf{F}$ is generated by a Brownian motion. Note that

$$dS_m(t) + dm(t) = d\left(\frac{\mathbf{E}\left[\int_0^1 g(s)dm(s)|\mathcal{F}_t\right]}{g(t)}\right).$$

Since a martingale can be represented by an Itô integral, the right-hand side is an Itô process by Itô's lemma as g is a strictly positive Itô process. Thus prices plus cumulative dividends of long-lived securities are Itô processes. These are all economically appealing properties of price processes of securities determined by preferences and information flow and independent of endowments and technologies.

8 Concluding Remarks

In this paper we advanced a family of topologies defining closeness between consumption patterns over time under uncertainty to capture the intuitive idea that consumptions at nearby dates are almost perfect substitutes. The idea we tried to formalize is certainly not new. Our contribution lies in defining the topologies and, more important, characterizing the topological dual spaces. We feel that our choice of topologies is appropriate for the following two reasons. First, in the degenerate case where the true state of nature is revealed at time 0, our conclusion agrees with the results under certainty of H&K. Second, in the nondegenerate case, the topological duals are natural generalizations of those under certainty. In this case, shadow prices of consumption are processes that can be decomposed as the sum of two components: a process of absolutely continuous sample path and a martingale. In the case of uncertainty a new element,

the pattern of information flow, affects the sample path properties of the shadow price process of consumption. This effect is captured in the martingale component of the process. It is known that a martingale can make discontinuous changes only at surprises. Thus the shadow prices for consumption are continuous except possibly at surprises. This is an intuitively appealing result which holds regardless of the nature of the endowment process.

The proposed family of topologies, however, does not give rise to certain mathematical properties known to be sufficient for the existence of an Arrow-Debreu equilibrium. We hope that further research produces more powerful existence theorems which can accommodate our model.

In all of our analysis, the information structure is assumed to be quasi left-continuous. This was made to simplify our exposition since then an anticipated event is synonymous to a predictable optional time. However, this assumption rules out important examples of information release such as announcements of macroeconomic news. Often, it is known to the public that some changes of macroeconomic policies will be announced at a deterministic time t although the details of the changes will not be known till the announcement. Then $\mathcal{F}_{t-} \neq \mathcal{F}_t$. Since any deterministic time is predictable, the information structure is thus not quasi left-continuous.

When the information structure is not quasi left-continuous, Propositions 1, 2, and 3 remain true as their proofs are independent of the special properties of the information structure. This may cause some confusion for the reader at first. For example, Proposition 2 implies that one unit of consumption for sure at t is a close substitute to one unit of consumption for sure at $t - \frac{1}{n}$ for large n even if there is some announcement of policy changes at time t and the details of this announcement are not known till t. One can explain this intuitively by observing that instants before t, the "average" or "expected" content of the announcement will be known. For example, market participants might anticipate tightening of money supply, but the exact amount of the new money supply is not known.

The actual announcement at time t could be either high reduction in money supply, which is perceived to be relatively "bad" news, or a low reduction in money supply, which is relatively "good" news given the anticipations instants before t. One unit of consumption at time t in the event of good news may be preferred to one unit of consumption instants before, which in turn may be preferred to one unit of consumption at t in the event of bad news. Considering the possibility of both "bad" and "good" news, agents find on the balance that one unit of consumption at t is a close substitute for one unit of consumption at time $t - \frac{1}{n}$, for large n, even when there are unanticipated events at t. The same intuition carries over to a general

predictable optional time T with an announcing sequence $\{T_n\}$. Here one unit of consumption at T is a close substitute to one unit of consumption at T_n for large n even if $\mathcal{F}_T \neq \bigvee_n \mathcal{F}_{T_n}$.

The general duality results remain valid. Propositions 4, 5, and 6 are also independent of the continuity property of the information structure. So the topological dual of $(\mathcal{E}^{\varphi}, \mathcal{T}_{\varphi})$ is essentially composed of processes that are the sum of processes with absolutely continuous sample paths and martingales. What is different here is the sample path properties of martingales. The discontinuity of a martingale can now happen either at non-predictable optional times or at predictable optional times where the information structure is not quasi left-continuous. These two kinds of optional times are times of "surprises." The duality results also help us understand the discussion above about the "good" and "bad" news. Suppose that the prices are represented by $g \in \mathcal{E}^{\varphi^*}$ with $g(t) = f(t) + m(t)$, where f is a process with absolutely continuous sample paths and m is a martingale. Let T be a predictable optional time with an announcing sequence $\{T_n\}$ such that $\mathcal{F}_T \neq \bigvee_n \mathcal{F}_{T_n}$ and suppose that $m(T) \neq lim_n m(T_n)$ with a strictly positive probability. Since one unit of consumption at T is a close substitute to one unit of consumption at T_n for large n, we expect their time zero price to be close despite the discontinuity of m at T. The time zero values of one unit of consumption at T_n and at time T are, respectively,

$$E[f(T_n) + m(T_n)] = E[f(T_n)] + m(0)$$

and

$$E[f(T) + m(T)] = E[f(T)] + m(0).$$

Closeness of the shadow prices is a consequence of the fact that m is a martingale and f has continuous paths. The effects of the discontinuities of m that correspond to "surprises" cancel each other out in expectations.

Similarly, in the arbitrage pricing of financial assets, everything holds true until the characterizations of the sample path properties of the price processes of long-lived securities. We leave the generalization to the reader.

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Appendix A Proofs

PROOF OF PROPOSITION 1:

PROOF. It suffices to show that for any $\gamma > 0$

$$\mathbf{E}\left[\int_0^1 \varphi\Big(\gamma|x_n(t) - x(t)|\Big)\, dt + \varphi\Big(\gamma|x_n(1) - x(1)|\Big)\right] \to 0.$$

The assertion then follows from the Lebesgue convergence theorem and the fact that $\varphi(\omega, z)$ is continuous in z and $\varphi(\omega, 0) = 0$. ∎

PROOF OF PROPOSITION 2:

PROOF. First, we show that $k\chi_{[T,1]} \in \mathcal{E}_+^\varphi$ for all optional time T and all $k > 0$. Note that

$$\mathbf{E}\left[\int_0^1 \varphi(\gamma k\chi_{[T,1]}(t))dt\right] \leq \mathbf{E}\left[\varphi(\gamma k)\right] < \infty,$$

where the first inequality follows from the fact that φ is strictly increasing and the second inequality follows from the hypothesis that φ is integrable.

Second, the fact that $T + \frac{1}{n}$ is an optional time whenever T is can be seen by directly checking the definition:

$$\{\omega \in \Omega : T + \frac{1}{n} \leq t\} = \{\omega \in \Omega : T \leq t - \frac{1}{n}\} \in \mathcal{F}_{t-\frac{1}{n}} \subset \mathcal{F}_t \;\forall t \in [0, 1].$$

Third, the hypothesis that φ is integrable and Lebesgue convergence theorem imply that

$$\begin{aligned}\lim_{n\to\infty} \mathbf{E}\left[\int_0^1 \varphi(\gamma k(\chi_{[T,1]}(t) - \chi_{[T+\frac{1}{n},1]}(t)))dt\right] &= \mathbf{E}\left[\int_0^1 \lim_{n\to\infty} \varphi(\gamma k(\chi_{[T,1]}(t) - \chi_{[T+\frac{1}{n},1]}(t)))dt\right] \\ &= \mathbf{E}\left[\int_0^1 \varphi(\gamma k \lim_{n\to\infty}(\chi_{[T,1]}(t) - \chi_{[T+\frac{1}{n},1]}(t)))dt\right] = 0,\end{aligned}$$

where the second and the third equalities follow from the fact that $\varphi(\omega, z)$ is continuous in z and is zero at $z = 0$, respectively.

Fourth, suppose that $\{T_n\}$ is a sequence of optional times with $T_n \leq T$, and $T_n < T$ and $T_n < T_{n+1}$ on the set $\{T > 0\}$ and that $k\chi_{[T_n,1]} \to k\chi_{[T,1]}$ in $\|\cdot\|_\varphi$. This implies that

$$\lim_{n\to\infty} \mathbf{E}\left[\int_0^1 \varphi(\gamma k(\chi_{[T,1]}(t) - \chi_{[T_n,1]}(t)))dt\right] \to \infty$$

for all $\gamma > 0$. This implies that $k\chi_{[T_n,1]} \to k\chi_{[T,1]}$ in $P \times \lambda$-measure, and hence there exists a subsequence $\{T_{n_1}\}$ such that

$$\lim_{n_1\to\infty} \chi_{[T_{n_1},1]}(\omega, t) - \chi_{[T,1]}(\omega, t) = 0 \quad P \times \lambda - a.e.,$$

which implies that $T_{n_1} \to T$ $a.s.$ and thus T is predictable.

The proof for the last assertion follows from bounded convergence theorem. ∎

Proof of Proposition 3:

Proof. Note again that $\|x_n - x\|_\varphi \to 0$, if and only if for any $\gamma > 0$

$$E\left[\int_0^1 \varphi\Big(\gamma|x_n(t) - x(t)|\Big)\,dt + \varphi\Big(\gamma|x_n(1) - x(1)|\Big)\right] \to 0;$$

see Musielak (1983, Theorem 1.6). Note that if $\mathbf{E}\Big[\rho(x_n, x)\Big] \to 0$, then

$$P\Big(\Big\{\rho(x_n, x) \geq \frac{1}{m}\Big\}\Big) \to 0 \quad \text{as} \quad n \to \infty.$$

For any integer m, let $N(m)$ be such that $P\Big(\Big\{\rho(x_n, x) \geq \frac{1}{m}\Big\}\Big) < \frac{1}{m}$, for all $n > N(m)$.

Next write:

$$E\left[\int_0^1 \varphi\Big(\gamma|x_n(t) - x(t)|\Big)\,dt + \varphi\Big(\gamma|x_n(1) - x(1)|\Big)\right] = I_1^{m,n} + I_2^{m,n},$$

for $n > N(m)$, where

$$\begin{aligned} I_1^{m,n} &= \int_{\{\rho(x_n,x)\geq\frac{1}{m}\}}\left[\int_0^1 \varphi\Big(\gamma|x_n(t) - x(t)|\Big)\,dt + \varphi\Big(\gamma|x_n(1) - x(1)|\Big)\right] P(d\omega), \\ I_2^{m,n} &= \int_{\{\rho(x_n,x)<\frac{1}{m}\}}\left[\int_0^1 \varphi\Big(\gamma|x_n(t) - x(t)|\Big)\,dt + \varphi\Big(\gamma|x_n(1) - x(1)|\Big)\right] P(d\omega). \end{aligned}$$

We will show that both integrals converge to zero, as m (and hence n) $\to \infty$.

First consider $I_1^{m,n}$. Since

$$\int_0^1 \varphi\Big(\gamma|x_n(t) - x(t)|\Big)\,dt \leq \int_0^1 \varphi\Big(\gamma(x_n(1) + x(t))\Big)\,dt \leq \varphi\Big(\gamma(x_n(1) + x(1))\Big) \quad \forall\omega \in \Omega,$$

we have

$$\begin{aligned} \int_0^1 \varphi\Big(\gamma|x_n(t) - x(t)|\Big)\,dt + \varphi\Big(\gamma|x_n(1) - x(1)|\Big) &\leq 2\varphi\Big(\gamma(x_n(1) + x(1))\Big) \\ &\leq 2\varphi\Big(\gamma(K + 2x(1))\Big) \end{aligned}$$

But since $\mathbf{E}[\varphi(K)] < \infty$ and $x \in \mathcal{E}^\varphi$, it then follows that $I_1^{m,n} \to 0$ as $m, n \to \infty$, by Lebesgue dominated convergence theorem.

Now consider $I_2^{m,n}$. Note first that for $\omega \in \{\rho_\omega(x_n, x) < \frac{1}{m}\}$, we have

$$|x_n(1) - x(1)| < \frac{1}{m} \quad \text{or} \quad \varphi\Big(\gamma|x_n(1) - x(1)|\Big) < \varphi\Big(\frac{\gamma}{m}\Big).$$

Thus

$$\int_{\{\rho(x_n,x)<\frac{1}{m}\}} \varphi\Big(\gamma|x_n(1) - x(1)|\Big)\,P(d\omega) < \mathbf{E}\Big[\varphi\Big(\frac{\gamma}{m}\Big)\Big].$$

Next consider the integral part. For any $\omega \in \{\rho(x_n, x) < \frac{1}{m}\}$, we can bound the inside integral by:

$$\frac{1}{m}\sum_{k=1}^{m}\varphi\left(\gamma\left(\max\{x(\omega,\frac{k}{m}),x_n(\omega,\frac{k}{m})\} - \min\{x(\omega,\frac{(k-1)}{m}),x_n(\omega,\frac{(k-1)}{m})\}\right)\right).$$

But since for precisely those sample functions $\rho(x_n, x) < \frac{1}{m}$ we have

$$\max\{x(\omega,t),x_n(\omega,t)\} \le x(\omega,t+\frac{1}{m}) + \frac{1}{m}$$

and

$$\min\{x(\omega,t),x_n(\omega,t)\} \ge x(\omega,t-\frac{1}{m}) - \frac{1}{m},$$

so the summation above is bounded by:

$$\frac{1}{m}\sum_{k=1}^{m}\varphi\left(\gamma\left(x(\omega,\frac{(k+1)}{m}) - x(\omega,\frac{(k-2)}{m}) + \frac{2}{m}\right)\right).$$

Finally, as φ is convex, equal to zero at zero and increasing, this in turn is bounded by:

$$\frac{1}{m}\varphi\left(\gamma\sum_{k=1}^{m}\left[x(\omega,\frac{(k+1)}{m}) - x(\omega,\frac{(k-2)}{m}) + \frac{2}{m}\right]\right) \le \frac{1}{m}\varphi\Big(\gamma(3x(\omega,1)+2)\Big).$$

Therefore,

$$\int_{\{\rho(x_n,x)<\frac{1}{m}\}}\int_0^1 \varphi\Big(\gamma|x_n(t)-x(t)|\Big)\,dt\,P(d\omega) \le \frac{1}{m}\mathbf{E}\left[\varphi\Big(\gamma(3x(1)+2)\Big)\right].$$

$$I_2^{m,n} < \frac{1}{m}\mathbf{E}\left[\varphi\Big(\gamma(3x(1)+2)\Big)\right] + \mathbf{E}\left[\varphi\Big(\frac{\gamma}{m}\Big)\right],$$

and $I_2^{m,n} \to 0$ as $m,n \to \infty$, proving our result. ∎

Proof for Proposition 4:

Proof. Integration by parts path-by-path gives

$$\begin{aligned}\psi(x) &= \mathbf{E}\left[\textstyle\int_{0-}^{1} g(t)dx(t)\right]\\ &= \mathbf{E}\left[\textstyle\int_{0-}^{1} f(t)dx(t) + \int_{0-}^{1} m(t)dx(t)\right]\\ &= \mathbf{E}\left[f(1)x(1) - \textstyle\int_0^1 x(t)f'(t)dt + m(1)x(1)\right]\\ &= \mathbf{E}\left[-\textstyle\int_0^1 x(t)f'(t)dt - x(1)f'(1)\right],\end{aligned} \tag{18}$$

where f' denotes the derivative of f and we have used the fact that

$$\mathbf{E}[\int_{0-}^{1} m(t)dx(t)] = \mathbf{E}[\int_{0-}^{1} m(1)dx(t)] = \mathbf{E}[m(1)x(1)] \tag{19}$$

in the third equality (see Dellacherie and Meyer (1982, VI.57)).

Musielak (1983, Corollary 13.14, p.87) shows that (18) is a linear functional continuous in $\mathcal{T}_\varphi$ if $f' \in L^{\varphi^*}$. This is the sufficiency part.

Now consider the necessity part. Let $\psi : \mathcal{E}^\varphi$ be a $\mathcal{T}_\varphi$-continuous linear functional. By the Hahn-Banach theorem, ψ can be extended to be a continuous linear functional on the whole of E^φ, which is

$$\left\{x \in \mathbf{O} \colon E\left[\int_0^1 \varphi\Big(\gamma|x(t)|\Big)\,dt + \varphi\Big(\gamma|x(1)|\Big)\right] < \infty \quad \forall \gamma > 0\right\},$$

where $\mathbf{O}$ denotes the space of processes that are measurable with respect to the sigma-field on $\Omega \times \mathcal{F}$ that is generated by all the processes adapted to $\mathbf{F}$ that have right-continuous paths. Let Ψ denote this extension. We first show that $\Psi : E^\varphi \to \Re$ can be represented in the form:

$$\Psi(x) = \mathbf{E}\left[\int_0^1 x(t)y(t)dt + x(1)y(1)\right] \quad \forall x \in E^\varphi \tag{20}$$

for some $y \in L^{\varphi^*}$. For this, we first consider the notion of *modular convergence*: A sequence $\{x_n\} \in L^\varphi$ is said to converge in modular to x, if

$$\mathbf{E}\left[\int_0^1 \varphi\Big(\gamma|x_n(t) - x(t)|\Big)\,dt + \varphi\Big(\gamma|x_n(1) - x(1)|\Big)\right] \to 0 \quad \text{for some } \gamma > 0.$$

The sequence $\{x_n\} \in L^\varphi$ converges in norm to x, if and only if

$$\mathbf{E}\left[\int_0^1 \varphi\Big(\gamma|x_n(t) - x(t)|\Big)\,dt + \varphi\Big(\gamma|x_n(1) - x(1)|\Big)\right] \to 0 \quad \text{for all } \gamma > 0;$$

see Musielak (1983, Theorem 1.6, p.3). Norm convergence of x_n clearly implies modular convergence.

Musielak (1983, Theorem 13.17, p.88) shows that (20) is true under the additional restriction on φ that

$$\forall x_0 > 0 \quad \text{there exists} \quad c > 0 \quad \text{such that} \quad \frac{\varphi(\omega, x)}{x} \geq c \quad \text{for} \quad x \geq x_0 \quad \forall \omega \in \Omega.$$

Musielak (1983, Theorem 13.15, p.87), however, shows that this restriction is required to guarantee that a norm continuous linear functional on L^φ is also modular continuous. In our formulation, we do not need this restriction, since we do not deal with the generalized Orlicz space L^φ. We only consider the subspace E^φ, and on this subspace norm convergence and modular convergence are equivalent; Musielak (1983, 5.2.B, p.18). We therefore conclude that restricting our attention to E^φ allows us to relax the above described condition. The interested reader can easily verify this by consulting Musielak (1983, Theorems 13.15 and 13.17).

Now define

$$f(\omega, t) \equiv -\int_0^t y(\omega, s)ds \quad \forall t \in [0,1]\, \omega \in \Omega,$$

which is clearly adapted and path-wise absolutely continuous. Denoting the derivative of $f(\omega, t)$ with respect to t by $f'(\omega, t)$, it is clear that $f'(t) = -y(t)$. Reversing the arguments in deriving (18), we prove the necessity part. ∎

Proof of Proposition 6:

PROOF. Let φ be integrably asymptotically linear. We will first show that if x is an element of $\mathcal{E}$, then it is also an element of $\mathcal{E}^{\varphi}$. Note the following:

$$\begin{aligned} \|x\|_{\varphi} &= \mathbf{E}[\int_0^1 \varphi(|x(t)|)\chi_{\{|x(t)|\leq K\}}\,dt + \varphi(|x(1)|)\chi_{\{x(1)\leq K\}}] \\ &+ \mathbf{E}[\int_0^1 \varphi(|x(t)|)\chi_{\{|x(t)|\geq K\}}\,dt + \varphi(|x(1)|)\chi_{\{x(1)\geq K\}}] \\ &\leq 2\mathbf{E}\left[\varphi(K)\right] + \alpha\mathbf{E}\left[\int_0^1 |x(t)|dt + |x(1)|\right] + 2\epsilon < \infty. \end{aligned}$$

Hence x is an element of $\mathcal{E}^{\varphi}$.

Since $\mathcal{T}$ is weaker than $\mathcal{T}_{\varphi}$, we only need to show that convergence in $\mathcal{T}$ implies convergence in $\mathcal{T}_{\varphi}$. Consider a sequence of elements $\{x_n\}$ that converges to x in $\mathcal{T}$. This implies that $\{x_n\}$ converges in the product measure generated by P and Lebesgue measure to x. Consider now

$$\begin{aligned} \|x_n - x\|_{\varphi} &= \mathbf{E}[\int_0^1 \varphi(|x_n(t) - x(t)|)\chi_{\{|x_n(t)-x(t)|\leq K\}}\,dt + \varphi(|x_n(1) - x(1)|)\chi_{\{x_n(1)-x(1)\leq K\}}] \\ &+ \mathbf{E}[\int_0^1 \varphi(|x_n(t) - x(t)|)\chi_{\{|x_n(t)-x(t)|\geq K\}}\,dt + \varphi(|x_n(1) - x(1)|)\chi_{\{x_n(1)-x(1)\geq K\}}] \\ &\leq \mathbf{E}[\int_0^1 \varphi(|x_n(t) - x(t)|)\chi_{\{|x_n(t)-x(t)|\leq K\}}\,dt + \varphi(|x_n(1) - x(1)|)\chi_{\{x_n(1)-x(1)\leq K\}}] \\ &+ \alpha\mathbf{E}\left[\int_0^1 |x_n(t) - x(t)|dt + |x_n(1) - x(1)|\right] + 2\epsilon. \end{aligned}$$

The first term on the right-hand side of the inequality converges to zero by Lebesgue convergence theorem. The second term goes to zero as $n \to \infty$ by hypothesis. Since ϵ is arbitrarily small, $\|x_n - x\|_{\varphi} \to 0$ as $n \to \infty$. ∎

PROOF OF PROPOSITION 7:

PROOF. From Propositions 4, 5, and 6, we know that $g(t) = f(t) + m(t)$, where f is an absolutely continuous process and $m(t)$ is a martingale. The assertion then follows from the fact that a martingale adapted to a Brownian motion filtration can always be represented as an Itô integral (see Clark (1970, theorems 3 and 4)). ∎

PROOF OF PROPOSITION 8:

PROOF. Suppose that u is not linear. Then there exists two scalars $r, \hat{r}$ and $t \in [0, 1]$ such that $u((r + \hat{r})/2, t) \neq u(r, t)/2 + u(\hat{r}, t)/2$. Without loss of generality, assume that $u((r + \hat{r})/2, t) > u(r, t)/2 + u(\hat{r}, t)/2$. By joint continuity, there exists $\epsilon > 0$ and some interval I containing t such that $u((r + \hat{r})/2, s) - \epsilon > u(r, s)/2 + u(\hat{r}, s)/2$ for all $s \in I$. Consider a sequence of nonrandom absolutely continuous consumption patterns constructed as follows: Off of I, consume at rate 1 in each x_n. On I, subdivide I into $2n$ equal sized intervals, and consume at rate r on the even subintervals and $\hat{r}$ on the odd. This sequence of consumption patterns converges in the Prohorov metric to the consumption pattern x that has $x'(s) = 1$ off I and $x'(s) = (r + \hat{r})/2$ on I. By Proposition 3, $x_n \to x$ in $\mathcal{T}_{\varphi}$. But $U(x) > U(x_n) + \epsilon\lambda(I)$, where $\lambda(I)$ is the Lebesgue measure of I. Thus U is not continuous in $\mathcal{T}_{\varphi}$. ∎

Proof of Proposition 9:

Proof. Observe that

$$U(x) = \mathbf{E}\left[\int_0^1 \alpha(t)x'(t)dt + \int_0^1 \beta(t)dt\right] = \mathbf{E}\left[\int_0^1 \alpha(t)dx(t) + \int_0^1 \beta(t)dt\right].$$

Thus U is a linear functional plus a constant. Then the assertion follows directly from Proposition 4. ∎

Proof of Proposition 10:

Proof. Fix a topology $\mathcal{T}_\varphi$. Let x_n be a sequence of consumption plans that converge to x in $\mathcal{T}_\varphi$. Assume, without loss of generality, that $\sup_n \|X_n\|_\varphi < \infty$. We will first show that $u(\hat{x}_n, t)$ converge to $u(\hat{x}, t)$ in the product measure $\nu = P \times \lambda$, where P is the probability measure on $(\Omega, \mathcal{F})$, and λ is the Lebesgue measure on the Borel sigma-field on $[0, 1]$.

First observe that if $x_n \to x$ in $\mathcal{T}_\varphi$, then $x_n \to x$ in $\mathcal{T}$. Hence,

$$\lim_{n\to\infty} \mathbf{E}[\int_0^1 |x_n(t) - x(t)|\, dt + |x_n(1) - x(1)|] = 0. \tag{21}$$

This implies that x_n converge to x in the product measure ν, and that $\int_0^1 |x_n(t) - x(t)|\, dt$ converges to zero in P-measure. Therefore, for any m, we can find $N(m)$ such that $\nu\Big(\{(\omega, t): |x_n(\omega, t) - x(\omega, t)| > \frac{1}{m}\}\Big) < \frac{1}{m}$, and $P\Big[\int_0^1 |x_n(t) - x(t)|\, dt > \frac{1}{m}\Big] < \frac{1}{m}$, for all $n > N(m)$.

Now consider $\hat{x}_n(\omega, t) - \hat{x}(\omega, t) = \int_0^k \theta(\omega, s)\, dx_n(\omega, t-s) - dx(\omega, t-s)$. Integrating by parts, and using the assumption that $\theta(\omega, k)$ vanishes, we get:

$$\hat{x}_n(\omega, t) - \hat{x}(\omega, t) = -\Big(x_n(\omega, t) - x(\omega, t)\Big)\theta(\omega, 0) - \int_0^k \Big(x_n(\omega, t-s) - x(\omega, t-s)\Big)\theta_s(\omega, s)\, ds, \tag{22}$$

where $\theta_s(\omega, s)$ denotes the partial derivative of $\theta(\omega, s)$ with respect to s. But since both θ and its derivative are uniformly bounded over ω, the left-hand side in (22) is bounded by $M_1|x_n(\omega, t) - x(\omega, t)| + M_2[\int_0^1 |x_n(\omega, t) - x(\omega, t)|\, dt]$, where M_1 and M_2 are two constants.

From this , we can easily conclude that

$$\nu\Big(\{(\omega, t): |\hat{x}_n(\omega, t) - \hat{x}(\omega, t)| > \frac{1}{m}(M_1 + M_2)\}\Big) < \frac{2}{m} \quad \forall n > N(m). \tag{23}$$

Hence $\hat{x}_n$ converge in ν-measure to $\hat{x}$. Now consider the convergence of $u(\hat{x}_n)$. First, assume that u is uniformly continuous in $\hat{x}$. Since the felicity function u is state-independent, and uniformly continuous in $\hat{x}$, it then follows that if $|\hat{x}_n - \hat{x}| < c_1$ then $|u(\hat{x}_n) - u(\hat{x})| < \alpha c_1$, where α is independent of ω, and $\hat{x}$. We use this assumption of uniform continuity, together with (23) to conclude that $u(\hat{x}_n)$ converges in ν-measure to $u(\hat{x})$. By Jensen's inequality, we have $\sup_n \mathbf{E}[\int_0^1 u(\hat{x}_n)\, dt] \leq \sup_n u(\mathbf{E}[\int_0^1 \hat{x}_n\, dt]) < \infty$. It then follows by Lebesgue dominated convergence theorem that $U(x_n)$ converge to $U(x)$.

Next consider the case when u is merely continuous in $\hat{x}$. For every integer m, define $u_m(\hat{x}) = u(\hat{x})\chi_{[\frac{1}{m} \leq \hat{x} \leq m]}$. For a fixed m, the function u_m is uniformly continuous in $\hat{x}$, and hence the arguments in the previous paragraph show that $U_m(x_n)$ converge to $U_m(x)$, where

$U_m(x)$ is defined exactly as $U(x)$, except that u is substituted by u_m. But $u_m(\hat{x})$ converge monotonically to $u(\hat{x})$ as $m \to \infty$. Applying the monotone convergence theorem, we conclude that

$$\lim_{n\to\infty} U(x_n) = \lim_{m,n\to\infty} U_m(x_n) = \lim_{m\to\infty} U_m(x) = U(x).$$

This shows that the utility definition given by (15) gives preferences continuous in $\mathcal{T}_\varphi$. ▮

Proof of Proposition 11:

Proof. Suppose not. By joint continuity, there exists two scalars $r, \hat{r}$ and $t \in [0,1]$ such that $v((r+\hat{r})/2, z, t) \neq v(r, z, t)/2 + u(\hat{r}, z, t)/2$ for all values of z. Without loss of generality, assume that $v((r+\hat{r})/2, z, t) > v(r, z, t)/2 + v(\hat{r}, z, t)/2$. By joint continuity again, there exists $\epsilon > 0$ and some interval I containing t such that $v((r+\hat{r})/2, z, s) - \epsilon > v(r, z, s)/2 + v(\hat{r}, z, s)/2$ for all $s \in I$. Consider a sequence of nonrandom absolutely continuous consumption patterns constructed as follows: Off of I, consume at rate 1 in each x_n. On I, subdivide I into $2n$ equal sized intervals, and consume at rate r on the even subintervals and $\hat{r}$ on the odd. This sequence of consumption patterns converges in the Prohorov metric to the consumption pattern x that has $x'(s) = 1$ off I and $x'(s) = (r+\hat{r})/2$ on I. By Proposition 3, $x_n \to x$ in $\mathcal{T}_\varphi$. Proposition 10 shows that $\lim_{n\to\infty} \int_0^1 v(x', \hat{x}_n, t)\, dt = \int_0^1 v(x', \hat{x}, t)\, dt$. But for any $\hat{x}_n$, we have:

$$\int_0^1 v(x', \hat{x}_n, t)\, dt > \int_0^1 v(x'_n, \hat{x}_n, t)\, dt + \epsilon\lambda(I).$$

Taking limits of both sides as $n \to \infty$, we get:

$$\int_0^1 v(x', \hat{x}, t)\, dt = \lim_{n\to\infty} \int_0^1 v(x', \hat{x}_n, t)\, dt \geq \lim_{n\to\infty} \int_0^1 v(x'_n, \hat{x}_n, t)\, dt + \epsilon\lambda(I).$$

Hence $V(x) \geq \lim_{n\to\infty} V(x_n) + \epsilon\lambda(I)$, where $\lambda(I)$ is the Lebesgue measure of I. Thus V is not continuous in $\mathcal{T}_\varphi$. ▮

Proof of Proposition 12':

Proof. The sequence of sure consumption patterns $\chi_{[1/2,1]} - \chi_{[(n+1)/2n,1]}$ converges to zero by Proposition 3. But the positive part (or the negative part) obviously does not. ▮